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Korteweg–de Vries–Burgers equation and the Painlevé property

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Abstract. It is shown that the Korteweg-de Vries–Burgers' equation possesses the Painlevé property conditionally. Using an algorithmic approach a travelling wave solution is reproduced.

1. Introduction

We consider the Korteweg-de Vries-Burgers' (KdVB) equation

$$u_t + 2auu_x + bu_{xx} + cu_{xxx} = 0 \tag{1}$$

where a, b, c are constants. This equation drew attention when Johnson [1] used it to model nonlinear waves in an elastic tube with dispersion and dissipation. Wijngaarden also considered it [2], and exact travelling wave solutions have been found recently [3–7]. In a previous paper [8] we showed that there is essentially only one known exact solution to the KdVB equation. We obtained this travelling wave solution by partial use of a Painlevé analysis.

Here we point out that the KdVB equation possesses the Painlevé property conditionally. Furthermore, we reproduce the exact solution by an algorithmic procedure which exhaustively utilizes information obtained from the coefficients in the Painlevé expansion.

2. Painlevé analysis

Weiss *et al* [9] have shown how the integrability of a partial differential equation is related to the 'Painlevé property' of the equation. This property, which we shall refer to as the WTC-Painlevé property, can be summarized as follows: The dependent variable u is expressed as an infinite series

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^{j} \tag{2}$$

where α is a negative integer determined by comparing the lowest powers of ϕ corresponding to the nonlinear and linear terms in the differential equation. For the KdVB equation (1) we find $\alpha = -2$. A recurrence relation is obtained for determining the coefficients u_j and in the case of equation (1) this takes the form

$$cu_j(j+1)(j-4)(j-6)\phi_x^2 = h(\phi_t, \phi_x, \ldots, u_0, u_1, \ldots, u_{j-1}) \tag{3}$$

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where h is a nonlinear function. Resonances occur at $j=-1,4,6$ where the u_j are arbitrary. If the recurrence relations are consistently satisfied at the resonances then the differential equation is said to possess the WTC-Painlevé property. However, as we shall see shortly, the KdVB equation (1) does not have this property.

The recurrence relations (3) for the KdVB equation are explicitly

$$\begin{aligned} u_{j-3,t}&+(j-4)u_{j-2}\phi_t+2a\sum_{m=0}^{j}u_{j-m}[(m-2)u_m\phi_x+u_{m-1,x}] \\ &+b[(j-3)(j-4)u_{j-1}\phi_x^2+(j-4)(u_{j-2}\phi_{xx}+2u_{j-2,x}\phi_x)+u_{j-3,xx}] \\ &+c[(j-2)(j-3)(j-4)u_j\phi_x^3+(j-3)(j-4)(3u_{j-1}\phi_x\phi_{xx}+3u_{j-1,x}\phi_x^2) \\ &+(j-4)(3u_{j-2,x}\phi_{xx}+3u_{j-2,xx}\phi_x+u_{j-2}\phi_{xxx})+u_{j-3,xxx}] \\ &=0. \end{aligned} \tag{4}$$

The first four members give

$$j=0 \qquad u_0=-\frac{6c}{a}\phi_x^2 \tag{5}$$

$$j=1 \qquad u_1=\frac{6b}{5a}\phi_x+\frac{6c}{a}\phi_{xx} \tag{6}$$

$$j=2 \qquad u_2=\frac{1}{2a}\left[-\frac{\phi_t}{\phi_x}-4c\frac{\phi_{xxx}}{\phi_x}+3c\frac{\phi_{xx}^2}{\phi_x^2}-\frac{6b}{5}\frac{\phi_{xx}}{\phi_x}+\frac{b^2}{25c}\right] \tag{7}$$

$$\begin{aligned} j=3 \qquad & -2a\phi_x^2u_3+\left(2a\phi_{xx}+\frac{2ab}{5c}\phi_x\right)u_2+\frac{b}{5c}\phi_t+\phi_{xt}+\frac{b^2}{5c}\phi_{xx} \\ & +\frac{6b}{5}\phi_{xxx}+c\phi_{xxxx}=0 \end{aligned} \tag{8}$$

$$j=4 \qquad \frac{\partial}{\partial x}(\text{left side of (8)})=0,\ u_4 \text{ arbitrary.} \tag{9}$$

Rather than write out the relations at $j=5$ and $j=6$, we simply state that there is an incompatibility between them, i.e. the $j=6$ relation is not expressible in the form of a zero identity with u_4 and u_6 both arbitrary. Thus there is a breakdown in the condition for the WTC-Painlevé property at the resonance $j=6$.

However, we say that the KdVB equation possesses the Painlevé property conditionally in the sense that compatibility at the resonance $j=6$ can be achieved for particular ϕ functions. This is demonstrated most easily by employing the 'reduced ansatz' [9] $\phi=x-\psi(t)$. There is consistency at $j=6$ only if u_4 is no longer arbitrary but is a specific function of ϕ.

It has been known for some time that the KdVB equation is non-integrable in the sense that its spectral problem is non-existent; Feudel and Steudel [10] first pointed this out in 1985 by showing that the equation has no prolongation structure. (For a general discussion on the relation between integrability and prolongation, see e.g. Dodd and Fordy [11].) The Painlevé method we have employed to confirm non-integrability has the advantage of being easier to use and also indicates conditional integrability. It is to our knowledge the first such treatment of the KdVB equation.

3. Solution to the KdVB equation

In [8] we showed how to obtain the Jeffrey and Xu solution [4] to equation (1) by putting $u_1=0$ in equation (6), solving this for ϕ and then requiring $u_2=0$ in (7) to determine the arbitrary functions. We assumed $u_j=0$ for $j\geqslant 3$.

Here we adopt a slightly different tactic which can be used as an algorithm for finding travelling wave solutions. We start by setting $u_j=0$ for all $j\geqslant 1$, so that a solution to the differential equation takes the form

$$u=u_0\phi^{-2} \tag{10}$$

where u_0 is given by the first recurrence relation (5). Out of the remaining recurrence relations only three survive upon making use of $u_j=0$ ($j\geqslant 1$). Using (5) we write them as

$$b\phi_x+5c\phi_{xx}=0 \tag{11}$$

$$\phi_x\phi_t+5b\phi_x\phi_{xx}+7c\phi_x\phi_{xxx}+12c\phi_{xx}^2=0 \tag{12}$$

$$\phi_x\phi_{xt}+b\phi_x\phi_{xxx}+c\phi_x\phi_{xxxx}+b\phi_{xx}^2+3c\phi_{xx}\phi_{xxx}=0. \tag{13}$$

Equation (11) implies that a travelling wave solution is possible, and using equations (12) and (13) we obtain the solution in the form (10), where

$$\phi=\mathrm{e}^{\theta}+A \tag{14}$$

$$\theta=-\frac{b}{5c}x-\frac{6b^3}{125c^2}t+\eta \tag{15}$$

and η, A are arbitrary constants. If $A=0$, the trivial solution $u=$ constant is obtained. The particular form of solution given by Jeffrey and Xu [4] is recovered when $A=1$. We note that these two authors and others assumed there was a travelling wave solution to the KdVB equation; our approach shows that there must be such a solution.

Next we look for two non-zero terms in the series expansion (2), and thereby pick up a little more information. That is, we set $u_j=0$ for $j\geqslant 2$ but insist that $u_0\neq 0$ and $u_1\neq 0$. The solution to the differential equation is now of the form

$$u=u_0\phi^{-2}+u_1\phi^{-1}. \tag{16}$$

The recurrence relations (4) give u_0 and u_1 as before in (5) and (6), together with three more survivors:

$$25c\phi_x\phi_t-b^2\phi_x^2+30bc\phi_x\phi_{xx}+100c^2\phi_x\phi_{xxx}-75c^2\phi_{xx}^2=0 \tag{17}$$

$$5b\phi_x\phi_t+50c\phi_x\phi_{xt}+25c\phi_t\phi_{xx}+3b^2\phi_x\phi_{xx}+60bc\phi_x\phi_{xxx}+125c^2\phi_x\phi_{xxxx}+30bc\phi_{xx}^2 -50c^2\phi_{xx}\phi_{xxx}=0 \tag{18}$$

$$b\phi_{xt}+5c\phi_{xxt}+b^2\phi_{xxx}+6bc\phi_{xxxx}+5c^2\phi_{xxxxx}=0 \tag{19}$$

where we have substituted for u_0 and u_1 in terms of ϕ. From (17), (18) and (19) a travelling wave solution of the form (16) is obtained with

$$\phi=\mathrm{e}^{\theta}+A \tag{20}$$

$$\theta=\pm\frac{b}{5c}x-\frac{6b^3}{125c^2}t+\eta \tag{21}$$

where A and η are arbitrary constants. Note that the sign ambiguity on x gives two waves, in opposite directions. If $A=1$ we obtain the two forms of solution given by Jeffrey and Xu [4].

If, instead, we commenced by setting $u_j=0$ for $j\geqslant 3$ and require u_0, u_1 and u_2 to be all non-zero, the series solution to the differential equation is

$$u=u_0\phi^{-2}+u_1\phi^{-1}+u_2 \tag{22}$$

where u_0, u_1 and u_2 are given by (5), (6) and (7). Only three more of the recurrence relations (4) survive, which we write in the form

$$25c\phi_x\phi_t+(50acu_2-b^2)\phi_x^2+(30bc+100c^2)\phi_x\phi_{xx}-75c^2\phi_{xx}^2=0 \tag{23}$$

$$2abu_2\phi_x+b\phi_t+5c\phi_{xt}+(10acu_2+b^2)\phi_{xx}+6bc\phi_{xxx}+5c^2\phi_{xxxx}=0 \tag{24}$$

$$u_{2t}+2au_2u_{2x}+bu_{2xx}+cu_{2xxx}=0 \tag{25}$$

upon using expressions (5) and (6). A travelling wave solution represented by

$$\phi=\mathrm{e}^{\theta}+A \tag{26}$$

$$\theta=kx-\omega t+\eta \tag{27}$$

where A and η are arbitrary constants, is obtained from equations (23)–(25) if and only if

$$k=\pm\frac{b}{5c} \qquad \text{and} \qquad \omega=\pm\frac{2ab}{5c}u_2+\frac{6b^3}{125c^2}. \tag{28}$$

We note that u_2 is an arbitrary (non-zero) constant, as can be shown by substituting (22), (26), (27) and (28) into the KdVB equation. Choosing $A=1$ and $u_2=-6b^2/25ac$ reproduces the form of solution given by Xiong [3], McIntosh [5] and Samsonov [7].

A further iteration of this method yields nothing because the recurrence relations at $j=6$ require $u_3=0$. Thus we have obtained all the information about travelling wave solutions to the KdVB equation that this method will give.

Finally, we make the observation that if (22) is viewed as part of a Bäcklund transformation, only solutions differing by a constant from a known travelling wave solution can be found. That is, for the KdVB equation there appears to be no way of obtaining non-trivially new travelling wave solutions from the known one via a Bäcklund transformation.

Closing remark. The method used above can be applied to many PDEs. We write out the recurrence relations using the Painlevé series and require that $u_j=0$ for all $j\geqslant\beta$, where β is the smallest positive integer value which guarantees a non-constant solution u to the PDE. The method then boils down to solving a small system of PDEs which is particularly simple when travelling waves exist. The procedure is repeated systematically for successive initial values of β until the information is exhausted. A further paper which exemplifies this technique is in preparation.

References

[1] Johnson R S 1970 *J. Fluid Mech.* **42** 49
[2] Van Wijngaarden L 1972 *Ann. Rev. Fluid Mech.* **4** 369
[3] Xiong S L 1989 *Chinese Sci. Bull.* **34** 1158

[4] Jeffrey A and Xu S 1989 *Wave Motion* **11** 559
[5] McIntosh I 1990 *Phys. Lett.* **143A** 57
[6] Samsonov A M 1988 Existence and amplification of solitary strain waves in non-linearly elastic waveguides *Preprint 1259* A F Ioffe Physical Technical Institute, St Petersburg
[7] Samsonov A M and Sokurinskaya E V 1989 *Nonlinear Waves in Active Media (Research Reports in Physics)* ed J Engelbrecht (Berlin: Springer) p 99
[8] Vlieg-Hulstman M and Halford W D 1991 *Wave Motion* **14** 267
[9] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[10] Feudel F and Steudel H 1985 *Phys. Lett.* **107A** 5
[11] Dodd R K and Fordy A P 1983 *Proc. R. Soc.* A **385** 389